(a) \( \bar{u}_1 = 0, \bar{v}_1 = 0 \), always exists;

(b) \( \bar{u}_2 = 0, \bar{v}_2 = M_v \), always exists (\( M_v \) is a carrying capacity for \( v \));

(c) \( \bar{u}_3 = M_u, \bar{v}_3 = 0 \), always exists (\( M_u \) is a carrying capacity for \( u \));

(d) Non-trivial steady state,

\[
\bar{u}_4 = \frac{N_u M_u (M_v - N_v)}{M_u M_v - N_u N_v}, \quad \bar{v}_4 = \frac{N_v M_v (M_u - N_u)}{M_u M_v - N_u N_v},
\]

is the intersection of the straight lines,

\[
1 - u/M_u - v/N_v = 0 \quad \text{and} \quad 1 - v/M_v - u/N_u = 0,
\]

that must occur in the first quadrant due to a natural biological constraint: \( \bar{u} > 0, \bar{v} > 0 \). Evidently, this steady state does not always exist since two straight lines may not necessarily intersect in the first quadrant!

Conditions for existence of steady state (d): we must have simultaneously

\[
M_v > N_v, \quad M_u > N_u
\]

or

\[
M_v < N_v, \quad M_u < N_u.
\]
Thus, we arrive at the following 4 different possible cases:

\begin{align*}
(1) & \quad M_v < N_v, \ M_u > N_u \\
(2) & \quad M_v > N_v, \ M_u < N_u \\
(3) & \quad M_v > N_v, \ M_u > N_u \\
(4) & \quad M_v < N_v, \ M_u < N_u
\end{align*}

**Characterization of steady states.** The Jacobian matrix:

\[
A(\bar{u}, \bar{v}) = \begin{pmatrix}
    k_1 \left[ 1 - \frac{2\bar{u}}{M_u} - \frac{\bar{v}}{N_v} \right] & -k_1 \frac{\bar{u}}{N_v} \\
    -k_2 \frac{\bar{v}}{N_u} & k_2 \left[ 1 - \frac{2\bar{v}}{M_v} - \frac{\bar{u}}{N_u} \right]
\end{pmatrix}.
\]
Let us evaluate the Jacobian matrix at various steady states:

(a) For \((\bar{u}_1, \bar{v}_1) = (0, 0)\),

\[
A(0, 0) = \begin{pmatrix}
    k_1 & 0 \\
    0 & k_2
\end{pmatrix}.
\]

Here \(\text{tr}A = k_1 + k_2 > 0\), \(\det A = k_1 \cdot k_2 > 0\), and \((\text{tr}A/2)^2 - \det A = (k_1 - k_2)^2/4 > 0\), and thus, this trivial steady state is always an \textbf{unstable node}.

Alternatively, we may notice that \(\lambda_1 = k_1 > 0\), \(\lambda_2 = k_2 > 0\) (i.e., \(\lambda\)'s are the elements standing on the main diagonal of \(A\) in the case where \(A\) is upper triangular, lower triangular, or diagonal matrix). We will use similar argument for the two steady states that we consider next.

(b) For \((\bar{u}_2, \bar{v}_2) = (0, M_v)\),

\[
A(0, M_v) = \begin{pmatrix}
    k_1 \left[1 - \frac{M_v}{N_v}\right] & 0 \\
    -k_2 \frac{M_v}{N_u} & -k_2
\end{pmatrix}.
\]

In the lower triangular matrix \(A\), the diagonal elements correspond to

\[\lambda_1 = k_1 \left[1 - \frac{M_v}{N_v}\right], \quad \lambda_2 = -k_2 < 0.\]

For \(M_v > N_v\), we have that \(\lambda_1 < 0\), and thus, in this case the steady state is a stable node.

For \(M_v < N_v\), we have that \(\lambda_1 > 0\), and thus, in this case the steady state is a saddle (unstable).

(c) For \((\bar{u}_3, \bar{v}_3) = (M_u, 0)\),

\[
A(M_u, 0) = \begin{pmatrix}
    -k_1 & -k_1 \frac{M_u}{N_v} \\
    0 & k_2 \left[1 - \frac{M_u}{N_u}\right]
\end{pmatrix}.
\]
In the upper triangular matrix $A$, the diagonal elements correspond to
\[ \lambda_1 = -k_1 < 0, \quad \lambda_2 = k_2 \left[ 1 - \frac{M_u}{N_u} \right]. \]

For $M_u > N_u$, we have that $\lambda_2 < 0$, and thus, in this case the steady state is a stable node.

For $M_u < N_u$, we have that $\lambda_2 > 0$, and thus, in this case the steady state is a saddle (unstable).

(d) For $(\bar{u}_4, \bar{v}_4)$, we have (see the system of equations from which we determined $\bar{u}_4 > 0, \bar{v}_4 > 0$):
\[
A(\bar{u}_4, \bar{v}_4) = \begin{pmatrix}
-k_1 \frac{\bar{u}_4}{M_u} & -k_1 \frac{\bar{u}_4}{N_u} \\
-k_2 \frac{\bar{v}_4}{N_u} & -k_2 \frac{\bar{v}_4}{M_v}
\end{pmatrix}.
\]

For the above matrix,
\[
\text{tr}A = -k_1 \frac{\bar{u}_4}{M_u} - k_2 \frac{\bar{v}_4}{M_v} < 0,
\]
\[
\det A = k_1 k_2 \frac{\bar{u}_4 \bar{v}_4}{M_u M_v} \left[ 1 - \frac{M_u M_v}{N_u N_v} \right].
\]

We recall that the steady state $(\bar{u}_4, \bar{v}_4)$ exists only if
\[ M_v > N_v, \quad M_u > N_u \]
or if
\[ M_v < N_v, \quad M_u < N_u. \]

In the former case $\det A < 0$, and so, the steady state is a saddle, while in the latter case $\det A > 0$, and the steady state is a stable node (it can be shown that $(\text{tr}A/2)^2 - \det A > 0$).
Let us now construct the phase portraits for various cases:

(1) $M_v < N_v$, $M_u > N_u$

(2) $M_v > N_v$, $M_u < N_u$

(3) $M_v > N_v$, $M_u > N_u$

(4) $M_v < N_v$, $M_u < N_u$

Principle of Competitive Exclusion!
11. SIR model: phase plane analysis. Let us repeat the schematic representation:

![SIR Diagram]

The system of equations for $S$, $I$, and $R$ (susceptible, infected, and recovered) has the form:

$$\frac{dS}{dt} = -\alpha IS,$$
$$\frac{dI}{dt} = +\alpha IS - \beta I,$$
$$\frac{dR}{dt} = +\beta I,$$

$S(0) = S_0$, $I(0) = I_0$, $R(0) = 0$ without immunization.

Here $\alpha > 0$ is the infection rate constant, $\beta > 0$ is the removal rate constant of infectives.

We recall that at any instant of time

$$S(t) + I(t) + R(t) = S_0 + I_0 = N = \text{const},$$

where $N$ is the total population (assume no deaths, births, etc.).

We also note that in the SIR system the equation for $R$ is decoupled from a closed system of 2 equations for $S$ and $I$ (and when $I$ is known, $R$ can be obtained by integration). Thus, for SIR model we analyze the system of 2 equations:

$$\frac{dS}{dt} = -\alpha IS,$$
$$\frac{dI}{dt} = (\alpha S - \beta)I.$$

The relation $S(t) + I(t) + R(t) = N$ means that the solutions of the above system will always lie inside the triangle $S(t) + I(t) \leq N$, $S(t) \geq 0$, $I(t) \geq 0$ (since $R \geq 0$), and, without immunization, the initial conditions will lie of the line $S_0 + I_0 = N$: 
Null-clines:

$$\frac{dS}{dt} = 0 : \quad I = 0; \quad S = 0;$$

$$\frac{dI}{dt} = 0 : \quad I = 0; \quad S = \frac{\beta}{\alpha} = \sigma.$$ 

**New feature!**: Here the steady states are not isolated points, but the whole $I = 0$, $S \geq 0$ half-line. This means that as $t \to \infty$ we have that $I(t) \to 0$, and $S(t) \to S(\infty) > 0 = \text{const}$ (that depends on initial conditions).

Let us characterize the signs of velocity vectors:

$$\frac{dS}{dt} < 0 : \quad \text{for all } I > 0, \quad S > 0.$$ 

$$\frac{dI}{dt} < 0 : \quad \text{for } I > 0, \quad S < \frac{\beta}{\alpha} = \sigma,$$ 

$$\frac{dI}{dt} > 0 : \quad \text{for } I > 0, \quad S > \frac{\beta}{\alpha} = \sigma.$$

We may have the following situations:
Some terminology: \( \sigma = \beta / \alpha \) is called relative removal rate; \( 1 / \sigma \) is the infection’s contact rate. The equation for \( I \) may be re-written as

\[
\frac{dI}{dt} = \beta \left( \frac{\alpha S}{\beta} - 1 \right) I.
\]

What happens at the beginning of the infection (at \( t = 0 \)) depends on the sign of the term in the brackets in the above equation, i.e., on whether

\[
\frac{\alpha S_0}{\beta} - 1 > 0 \quad \text{or} \quad \frac{\alpha S_0}{\beta} - 1 < 0.
\]

The quantity (common notation that has nothing to do with recovered!)

\[
R_0 = \frac{\alpha S_0}{\beta} \quad \text{or, if} \quad I_0 \ll N, \quad \text{so that} \quad S_0 \approx N, \quad R_0 = \frac{\alpha N}{\beta}
\]

is called the basic reproduction rate of the infection. This number shows how many secondary infections are produced from one primary infection in the initial population. Interpretation of the term \( 1 / \beta \): it is the characteristic time showing for how long the individual infection lasts. Evidently, if

\[
R_0 > 1,
\]

then \( dI/dt > 0 \), and we have an epidemic.

How do we define an epidemic (number of new infections is increasing, or via total fraction of the original population infected, etc.)?
If we assume that $dI/dt > 0$ corresponds to epidemic, and $dI/dt < 0$ corresponds to no epidemic, then we get the following implication for immunization policies: the fraction of population, $p$, that must be immunized (with fraction $(1 - p)$ not immunized) is defined by the relation

$$\frac{\alpha(1 - p)N}{\beta} = (1 - p)R_0 < 1,$$

and thus,

$$p > 1 - \frac{1}{R_0},$$

where $R_0$ is the original basic reproduction rate of the infection.

Why does the disease die out: from the lack of infectives or from the lack of susceptibles?

12. SIRS model: phase plane analysis. We recall the schematic representation for SIRS:
Corresponding model system has the form (here $1/\gamma$ is the characteristic time it takes for the recovered to become susceptible again):

$$\frac{dS}{dt} = -\alpha IS + \gamma R,$$
$$\frac{dI}{dt} = +\alpha IS - \beta I,$$
$$\frac{dR}{dt} = +\beta I - \gamma R,$$

$S(0) = S_0$, $I(0) = I_0$, $R(0) = 0$ without immunization.

Once again, we recall that the quantity $S(t) + I(t) + R(t) = N = S_0 + I_0 = \text{const}$ is conserved. Thus, we may express

$$R(t) = N - I(t) - S(t).$$

Substituting this into equations for $S$ and $I$, we obtain:

$$\frac{dS}{dt} = -\alpha IS + \gamma (N - I - S) = f(S, I),$$
$$\frac{dI}{dt} = +\alpha IS - \beta I = g(S, I),$$

$S(0) = S_0$, $I(0) = I_0$.

**Null-clines:**

$$\frac{dS}{dt} = 0 : -\alpha IS + \gamma (N - I - S) = 0;$$
$$\frac{dI}{dt} = 0 : I = 0; S = \frac{\beta}{\alpha} = \sigma.$$

The functional representation of $S$ null-cline can be written as

$$I = \frac{\gamma(N - S)}{\alpha S + \gamma}.$$

It follows from the above expression that this null-cline intersects the $S$- and $I$-axes at the points $(N,0)$ and $(0,N)$, respectively. Also, it is immediately seen that in the first quadrant this curve lies below the straight line $I = N - S$. 

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We have either one or two steady states (intersections of null-clines) in the first quadrant:
(a) \((\bar{S}_1, \bar{I}_1) = (N, 0)\),
(b) \((\bar{S}_2, \bar{I}_2) = \left(\frac{\beta}{\alpha}, \frac{\gamma(N - \beta/\alpha)}{\beta + \gamma}\right)\).

We note that the second (non-trivial) steady state exists only if the following condition is satisfied:
\[
\frac{\alpha N}{\beta} = R_0 > 1.
\]

Let us characterize the steady states when \(R_0 < 1\) and when \(R_0 > 1\). The Jacobian matrix has the form:
\[
A(\bar{S}, \bar{I}) = \begin{pmatrix}
-(\alpha \bar{I} + \gamma) & -(\alpha \bar{S} + \gamma) \\
\alpha \bar{I} & \alpha \bar{S} - \beta
\end{pmatrix}.
\]

For \(R_0 = \frac{\alpha N}{\beta} < 1\) only one steady state, \((\bar{S}_1, \bar{I}_1) = (N, 0)\), exists and
\[
A(\bar{S}_1, \bar{I}_1) = \begin{pmatrix}
-\gamma & -(\alpha N + \gamma) \\
0 & \alpha N - \beta
\end{pmatrix}.
\]

From the above (taking into account that the above matrix is upper triangular), we obtain:
\[
\lambda_1 = -\gamma < 0, \quad \lambda_2 = \alpha N - \beta < 0.
\]
Thus, this unique steady state is a stable node. The phase portrait in this case:

For $R_0 = \alpha N/\beta > 1$ we have two steady states.

(a) For $(\bar{S}_1, \bar{I}_1) = (N, 0)$ we have the same Jacobian matrix as in the previous case, but now (due to condition on $R_0$)

$$\lambda_1 = -\gamma < 0, \quad \lambda_2 = \alpha N - \beta > 0.$$ 

So, this steady state is a saddle.

(b) For $(\bar{S}_2, \bar{I}_2)$, with $\bar{S}_2 > 0$, $\bar{I}_2 > 0$, we obtain

$$A(\bar{S}_2, \bar{I}_2) = \begin{pmatrix} - (\alpha \bar{I}_2 + \gamma) & - (\beta + \gamma) \\ \alpha \bar{I}_2 & 0 \end{pmatrix}.$$ 

Since $\text{tr}A(\bar{S}_2, \bar{I}_2) = - (\alpha \bar{I}_2 + \gamma) < 0$, and $\det A(\bar{S}_2, \bar{I}_2) = \gamma \beta (R_0 - 1) > 0$, the non-trivial steady state is either stable node or stable focus (depending on the numerical values of parameters).
The phase portrait in the case of oscillatory approach to \((\tilde{S}_2, \tilde{I}_2)\):