
1. (a) Consider a population growth model with unlimited resources:

\[ \frac{dP}{dt} = kP, \quad (k > 0), \quad P(0) = P_0. \]

Express the doubling time (i.e., the time it takes for the originally given population to double) in terms of the growth rate constant \( k \).

(b) Now consider a population decay model (observed when “birth” rate constant is smaller than “death” rate constant):

\[ \frac{dP}{dt} = -kP, \quad (k > 0), \quad P(0) = P_0. \]

Express the half time (i.e., the time it takes for the originally given population to become twice smaller) in terms of the growth rate constant \( k \).

2. A more general population growth model is written in the form

\[ \frac{dP}{dt} = k(P) \cdot P, \quad P(0) = P_0, \]

where growth rate coefficient \( k(P) \) is a function of population size \( P \). For sample choices of \( k(P) \) shown below, determine for which models nontrivial steady states exist. Are these steady states stable or unstable? Note: all coefficients are positive constants; biologically meaningful population values \( P(t) \geq 0 \).

(a) \( k(P) = (P - A) \);
(b) \( k(P) = K \cdot \exp(-aP) \);
(c) \( k(P) = A - \exp(aP) \);
(d) \( k(P) = K/(A + P) \);
(a) \( k(P) = (A - P) \).

3. Given a model equation

\[ \frac{du}{dt} = u(1 - u)(2 - u). \]

(a) Find the steady states of this equation.
(b) Which of the steady states are stable/unstable.
(c) Specify the domains of attraction of the stable steady states.
(d) Consider a solution $u(t)$ of the above differential equation with initial condition

$$u(0) = 1.5.$$ 

Find $\lim_{t \to \infty} u(t)$.

4. Modify the MATLAB programs presented on the course web site appropriately to solve numerically the following hypothetical population growth equation

$$\frac{du}{dt} = \cos(u) \cdot u,$$

with initial conditions:

(a) $u(0) = 1$
(b) $u(0) = 3$
(c) $u(0) = 5$
(d) any other initial condition(s) of your choice.

Plot your results in one graph. Can you make any conclusions on qualitative behavior of model solutions based only on your numerical computations? (For example, which values will solutions approach as $t \to \infty$?)

Next, find possible steady states of the above equation analytically. How many stable steady states are there? Is it possible to find all these steady states numerically solving model equation with various initial conditions for long time intervals (without theoretical analysis)?

5. [Graduate students only.] Consider logistic equation

$$\frac{du}{dt} = K \cdot u \left(1 - \frac{u}{M}\right).$$

The shapes of solution curves of this equation depend on the choice of initial condition. For small initial populations, $u(0) = u_0 \ll M$, the solution curves have two distinct parts, concave up (accelerated growth) and concave down (decelerated growth). For concave up part of solution curve the following inequality must be satisfied: $d^2u/dt^2 > 0$, while for concave down part of solution curve the following inequality must be satisfied: $d^2u/dt^2 < 0$.

(a) Show that solution curve is concave up for $u_0 < u < M/2$ and concave down for $M/2 < u < M$.
(b) Show that solution curve is concave up if initial condition $u_0 > M$.  

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