PART III:
Beyond secondary creep: anisotropic flow laws and the theory of continuous diversity

1. Fabric and its evolution: Available anisotropic flow laws

2. Continuous diversity of polycrystalline ice masses: the most comprehensive theory to model induced anisotropy
### 1. Fabric and its evolution: Available anisotropic laws

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<td>Morland &amp; Staroszczyk 1998~2001 obtain evolving anisotropy from instantaneous states of deformation without explicit reference to fabric or grain size (! reversibility of anisotropy)</td>
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1. Fabric and its evolution: Available anisotropic laws

H – (homogenization) models: based on averages of individual grains

Schmidt tensors: \( \dot{\Sigma}_{ij} = \frac{1}{N} \sum_{g=1}^{n} m_i^{(g)} c_j^{(g)} \)

- \( m_i^{(g)} \): unit vector parallel to resolved shear stress in the basal plane
- \( c_j^{(g)} \): unit vector parallel to the c-axis orientation
- \( n \): Number of grains \( g \)

S – (statistical) models: based on an Orientation Distribution Function

ODF: orientation density \( f=f(x,t,n) \), \( n \) vector of unit length in \( S^2 \)

\[
\int_{S^2} f(x,t,n) \, d^2n = 1
\]

alignment/structure/anisotropy tensors: \( A := \int_{S^2} f(x,t,n) \, n \otimes n \, d^2n \)
2. Continuous diversity of polycrystalline ice masses: the most comprehensive theory to model induced anisotropy

The **theory of mixtures with continuous diversity** (MCD)

- has been developed by S. H. Faria from ~ 2001
- conforms to the principles of Rational Mechanics
- is a thermodynamic theory
- is the most comprehensive theory to model heterogeneity, in particular induced anisotropy
- has many other applications

In the context of ice sheet modeling, MCD allows for the simultaneous modeling of

- texture evolution (rotation of c-axis)
- recrystallization, polygonization, recovery
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Background to the MCD:

Single constituent continua:

5 scalar balance laws for independent primary fields \( \rho(x,t), \mathbf{v}(x,t), T(x,t) \)

General balance law:

\[
\frac{\partial (*)}{\partial t} + \operatorname{div} [ (*) \mathbf{v} + \phi] - s = p
\]

\((*)\): additive quantity, \(\mathbf{v}\): velocity, \(\phi, s, p\): flux, supply and production of \((*)\)

\(\operatorname{div}\): divergence operator in Euclidean space \(E^3\)

Conservation equation: \(p=0\)

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Multiconstituent continua (chemically reacting mixtures, granular media,...):

N discrete constituents, indexed by $\alpha$, typically $N \leq 3$

5N balance laws for primary fields: $\rho_\alpha(x,t), v_\alpha(x,t), T_\alpha(x,t)$ for $\alpha = 1, \ldots, N$

General balance law:

$$\frac{\partial (\rho_\alpha)}{\partial t} + \text{div} [ (\rho_\alpha) v_\alpha + \phi_\alpha ] = -s_\alpha = p_\alpha$$

Non-conservation equations on constituent level: $p_\alpha \neq 0$

Mixture balance laws are derived from the constituent balance laws according to the Rational Mechanics Modeling of Materials approach (Truesdell's third metaphysical principle) and provide homogenization rules:

$$\sum_{\alpha=1}^N \rho_\alpha = \rho \quad \sum_{\alpha=1}^N T_\alpha = \rho_\alpha \cdot \mathbf{u}_\alpha \otimes \mathbf{u}_\alpha = \mathbf{T} \quad \mathbf{u}_\alpha = \mathbf{v}_\alpha - \mathbf{v} \text{ diffusion velocity}$$
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Classical continuous mixtures

Countable set of constituents with individual primary variables, e.g. \( \rho_\alpha(x, t) \)

Each constituent has countably many properties distinguishing it from other constituents

Nature shows us often the reverse situation:

Mixtures with continuous diversity

Infinitely many constituents with primary variables depending on the continuously varying species label

\( \rho^*(x, t, \alpha) \)

\( \alpha \) in \( A = [\alpha_{\min}, \alpha_{\max}] \) species assemblage

 Constituents differ from each other only in very few properties (size, orientation, age, ...)

Polycrystalline ice
2. Continuous diversity of polycrystalline ice masses: the most comprehensive theory to model induced anisotropy

Mixtures with continuous diversity

Balance equations for primary fields (note: \# does not increase with $\alpha$) depend on position in i/ Euclidean space and ii/ Species space

$$\rho(x,t,\alpha), \mathbf{v}(x,t,\alpha), T(x,t,\alpha) \quad \alpha \text{ in } A = [\alpha_{\text{min}}, \alpha_{\text{max}}]$$
2. Continuous diversity of polycrystalline ice masses: the most comprehensive theory to model induced anisotropy

Mixtures with continuous diversity

Balance equations for primary fields (note: # does not increase with $\alpha$) depend on position in i/ Euclidean space and ii/ Species space

$$\rho(x, t, \alpha), \mathbf{v}(x, t, \alpha), T(x, t, \alpha) \quad \alpha \in \mathcal{A} = [\alpha_{\min}, \alpha_{\max}]$$

Polycrystalline ice: species (single crystals) are identified by their orientation, represented by a unit normal vector $\mathbf{n}$ in $S^2$
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Mixtures with continuous diversity

Balance equations for primary fields (note: # does not increase with α) depend on position in i/ Euclidean space and ii/ Species space

\[ \rho(x, t, \alpha), v(x, t, \alpha), T(x, t, \alpha) \quad \alpha \text{ in } A = [\alpha_{\min}, \alpha_{\max}] \]

Primary fields amended by dislocation density \( \rho_D \) and c-axis spin velocity \( s \):

\[ \rho(x, t, n), \rho_D(x, t, n), s(x, t, n), v(x, t, n), T(x, t, n) \]

General Balance Equation

\[ \partial (\cdot) / \partial t + \text{div}_{E_3} [(\cdot)v + \phi] + \text{div}_{S_2} [(\cdot)w + \psi] - s = p \]

\( w \) interspecies transition rate \quad \psi \) interspecies flux

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Species balance equation for polycrystals modeled as mixtures with continuous diversity (Faria, 2006, Proc. R. Soc. Lond. A)

Balance of mass: includes recrystallization
Balance of dislocation density: includes interspecies flux density of dislocations and production rate of dislocations
Balance of linear momentum: includes interspecies stress and high-angle interaction force
Balance of lattice spin velocity: includes polygonization tensor (interspecies couple stress) and high-angle interaction couple
Balance of internal energy: includes dissipative contributions associated with all new interspecies quantities
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Homogenization of species balance equations:

explores Rational Mechanics Modeling of Materials approach, is of type

$$\rho = \int_{S^2} \rho(x, t, n) \, d^2n$$

$$T = \int_{S^2} \left( T(x, t, n) - \rho(x, t, n) \left[ v(x, t, n) - v(x, t) \right] \otimes \left[ v(x, t, n) - v(x, t) \right] \right) \, d^2n$$

continuous mixtures:

$$\sum_{\alpha=1}^{N} \rho_{\alpha} = \rho$$

$$\sum_{\alpha=1}^{N} \mathbf{T}_{\alpha} - \rho_{\alpha} \mathbf{u}_{\alpha} \otimes \mathbf{u}_{\alpha} = \mathbf{T}$$
2. Continuous diversity of polycrystalline ice masses: the most comprehensive theory to model induced anisotropy

Constitutive theory:
Work in progress

In Part III: Simplified reduced model
2. Continuous diversity of polycrystalline ice masses: the most comprehensive theory to model induced anisotropy

The simplified model presented in Faria's Part III is still more general than all other anisotropic flow laws.

\[ T = -\rho \ 1 + \mu^{(4)} D^E \quad \text{with} \quad \mu^{(4)} = \mu^{(4)}(\rho_D, n, ...) \]

It encompasses the previously suggested models by

- Svendsen/Gödert/Hutter
- Azuma/Goto–Azuma
2. Continuous diversity of polycrystalline ice masses: the most comprehensive theory to model induced anisotropy

The flow law in the CAFFE model:

\[ D = E(T^D, A^{(2)}, B^{(4)}) A(T) \sigma^{n-1} T^D \]

\[ A^{(2)} = \int_{S^2} f(x, t, n) \mathbf{n} \times \mathbf{n} \, d^2n \]

\[ B^{(4)} = \int_{S^2} \mathbf{n} \times \mathbf{n} f(x, t, n) \mathbf{n} \times \mathbf{n} \, d^2n \]

The CAFFE model is implemented in Elmer/Ice @ CSC Finland (Th. Zwinger)